

Applications of Metric Spaces in Computational Complexity Theory

Research Article

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Abstract: Computational complexity theory heavily relies on b -dislocated metric spaces, quasi-metric spaces, and partial metric spaces for structuring algorithms and evaluating their efficiency. These mathematical frameworks contribute to performance evaluation, ensuring effective problem classification and optimization techniques. This paper investigates how these metric spaces contribute to Computational complexity theoretical aspects such as algorithm efficiency, complexity analysis and other applications. For example, we show that the solution to the divide and conquer recurrence equation is unique regardless of the size of a natural number used to split the problem and the complexity distance function can be used to provide asymptotic upper bound of the divide and conquer recurrence. In addition, we use b -relaxed triangle inequality of b -dislocated metric to measure similarity between documents, videos and pictures.

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Keywords: Metric Space • Quasi-Metric Space • Partial Metric Space • b -dislocated metric • Complexity theory • Fixed Point • Algorithm

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1. Introduction

Theoretical computer science has seen immense growth, with metric spaces serving as a backbone for various computational advancements. These spaces provide a structured mathematical framework that allows researchers to study distances and topologies within algorithms. The application of metric spaces manifests in numerous fields, such as clustering algorithms, nearest-neighbor searches, dimensionality reduction, and complexity theory. By leveraging metric-based principles, computer scientists can optimize data processing techniques, enhancing computational efficiency and algorithmic performance.

Metric spaces are defined [1] formally as a pair (X, d) , where X is a nonempty set and $d : X \times X \rightarrow \mathbb{R}^+$ satisfies:

1. $d(x, y) \geq 0, \forall x, y \in X$, (Non – negativity)

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2. $d(x, y) = 0 \Leftrightarrow x = y$, (*Equality*)
3. $d(x, y) = d(y, x)$, (*Symmetry*)
4. $d(x, z) \leq d(x, y) + d(y, z)$, (*Triangular Inequality*).

Beyond classical metric spaces, quasi-metric spaces and partial metric spaces extend their functionality in theoretical computing. A quasi-metric space (X, d) modifies the symmetry requirement of traditional metric spaces, allowing:

$$d(x, y) \neq d(y, x) \text{ and } d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X.$$

This asymmetry is useful in applications such as approximation algorithms and computational order structures.

Similarly, partial metric spaces refine standard metrics by permitting nonzero self-distances [2], making them valuable in program verification and computational analysis. A partial metric space (X, p) satisfies:

$$p(x, x) \geq 0, \quad p(x, x) \leq p(x, y) \text{ and } p(x, y) = p(y, x).$$

These modifications enhance fixed-point theory and complexity analysis of recursive algorithms.

Another key component of complexity theory is data structures, which organize and store information efficiently. The complexity of these data structures often relies on metric-based optimizations for improving computational speed and resource management.

Computational complexity theory provides a framework for analyzing the efficiency of algorithms by classifying problems based on their solvability. Metric spaces contribute to complexity theory by offering techniques for evaluating asymptotic performance. The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ extends these principles, where $d_{\mathcal{C}}(f, g)$ is defined as:

$$d_{\mathcal{C}}(f, g) = \sup \left(\frac{f(n)}{g(n)}, 0 \right),$$

characterizing how function growth rates determine algorithmic efficiency.

Beyond traditional computing applications, computer forensics relies on metric spaces for investigative processes such as digital evidence comparison and cryptographic security. b -dislocated metric spaces enhance forensic analysis by refining distance measurement in image recognition, fingerprint matching, and document verification, using the relation:

$$d(x, z) \leq s[d(x, y) + d(y, z)], \quad s \geq 1.$$

This flexibility allows for efficient classification of digital artifacts in security-sensitive environments.

Metric spaces serve as indispensable tools in algorithm development, data structuring and complexity theory. Their continued exploration offers new methods for improving computational efficiency, advancing forensic investigations, and optimizing real-world applications.

2. Links to the literature

Complexity theory has witnessed significant advancements in recent years, with metric spaces playing a vital role in these developments. For instance, metric spaces have far-reaching implications for computational complexity theory, particularly in the context of approximation algorithms.

In 1906, Fréchet [3] introduced the notion of metric spaces. At first, he did not call it a metric due to Hausdorff, one of the founding father of modern topology. In 1994, Matthews [5], introduced the notion of a partial metric space in order to obtain a suitable mathematical tool for program verification. He defined a partial metric space as a non-empty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ such that $\forall x, y, z \in X$:

1. $p(x, x) = p(x, y) = p(y, y) \Leftrightarrow x = y$;
2. $p(x, x) \leq p(x, y)$;
3. $p(x, y) = p(y, x)$;
4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A pair (X, p) such that X is a non-empty set and p is a partial metric on X is called a partial metric space.

In 1995, Matthews [6] extended Banach's fixed point theorem to the framework of partial metric spaces in view of its applications to program verification to formulate a suitable test for lazy data flow deadlock in Kahn's model of parallel computation.

We state the following proposition and theorems from [6–8, 12, 13].

Theorem 2.1.

For each complete Partial Metric $p : U^2 \rightarrow \mathbb{R}$, and for each function $f : U \rightarrow U$ such that,

$$\exists 0 \leq c < 1, \quad \forall x, y \in U, \quad p(f(x), f(y)) \leq c p(x, y)$$

called a contraction, firstly there exists a unique $a \in U$ such that $a = f(a)$, and secondary $p(a, a) = 0$.

In 1995, Schellekens [8, page 5-6] introduced a topological foundation for the asymptotic complexity analysis of algorithms.

In 2004, Oltra and Valero [12] defined a quasi-metric space as a non-negative real-valued function d on $X \times X$ such that $\forall x, y, z \in X$;

1. $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,
2. $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X, d) such that X is a non-empty set and d is a quasi-metric on X , is called a Quasi-Metric Space.

In 2022, T.K. Dey and Y. Wang [13] defined a topology as a collection $\mathcal{T}$ of subsets of a non-empty set X satisfying the conditions (1) Closure under set inclusion, (2) Closure under countable unions of sets and (3) Closure under finite intersection of sets.

The next result reveals the reason for which the partial metric spaces can be used as a suitable mathematical tool to describe Scott processes.

Proposition 2.1.

Given a complete partial metric space (X, p) . The partial metric topology $\tau(p)$ is a Scott-like topology over $(X, \leq_p)$, meaning every increasing sequence $(x_n)_{n \in \mathbb{N}}$ in $(X, \leq_p)$ has least upper bound and converges to it with respect to $\tau(p)$.

Later Matthews extended Banach's fixed point theorem to the framework of partial metric spaces in view of its applications to program verification [5] to formulate a suitable test for lazy data flow deadlock in Kahn's model of parallel computation [6].

Theorem 2.2.

Let f be a mapping of a complete partial metric space (X, p) into itself such that there is $s \in [0, 1]$ such that

$$p(f(x), f(y)) \leq s p(x, y)$$

$\forall x, y \in X$. Then f has a unique fixed point. Moreover, if $x^* \in X$ is the fixed point of f , then $p(x^*, x^*) = 0$.

Every partial metric induces a metric [5]. Given a partial metric space (X, p) , then the function $p^s : X \times X \rightarrow \mathbb{R}^+$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

$\forall x, y \in X$ is a metric on X . A sequence $(x_n)_{n \in \mathbb{N}}$ in (X, p) converges to $x \in X$ with respect to $\tau(p^s)$ iff $\lim_{n \rightarrow \infty} p(x, x_n) = \lim_{n \rightarrow \infty} p(x_n, x_n) = p(x, x)$; it is Cauchy iff it is Cauchy in the metric space (X, p^s) , and (X, p) is complete iff (X, p^s) is complete.

Since the establishment of theorem 2.2, there has been various intense research on fixed point results in partial metric spaces.

Theorem 2.3.

Suppose f is a mapping of a complete dualistic partial metric space (X, p) into itself such that there is a real number r with $0 \leq r < 1$, satisfying

$$|p(f(x), f(y))| \leq r |p(x, y)|,$$

for all $x, y \in X$. Then f has a unique fixed point.

Suppose $C \subseteq \mathcal{C}$, the a functional $\Phi : C \rightarrow C$ is an improver with respect to a function $f \in C$ provided Φ is monotone and that $\Phi(f)(n) \leq_{\infty} f(n)$ for all $n \in \mathbb{N}$. The following result was stated in [8];

Theorem 2.4.

The Divide and Conquer recurrence eq. (3) has a unique solution f_T in $\mathcal{C}$. Moreover, if the monotone functional Φ_T associated to eq. (3), and given by eq. (4), is an improver with respect to some function $g \in \mathcal{C}_{b,c}$ then the solution to the recurrence Equation satisfies that $f_T \in \mathcal{O}(g)$.

Schellekens [8] discusses the complexity class of Mergesort in order to illustrate the utility of theorem 2.4. The Mergesort running time of computing satisfies the following particular case of the recurrence eq. (3):

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(\frac{n}{2}) + \frac{n}{2} & \text{if } n \in \mathbb{N}_2, \end{cases} \quad (1)$$

Observe that theorem 2.4 provides that the recurrence eq. (1) has a unique solution $f_T^M \in \mathcal{C}_{2,c}$. We can easily check that the functional Φ_T given by eq. (4) and induced by the recurrence eq. (1) is an improver with respect to the complexity function $g_k \in \mathcal{C}_{2,c}$ with $k > 0$, if and only if $k \leq \frac{1}{2}$, where

$$g_k(n) = \begin{cases} c & \text{if } n = 1 \\ kn \log_2(n) & \text{if } n \in \mathbb{N}_2, \end{cases} \quad (2)$$

By theorem 2.4, we have that $f_T^M \in \mathcal{O}(g_{\frac{1}{2}})$. Thus the complexity function $g_{\frac{1}{2}}$, gives an asymptotic upper bound of the running time of computing of the mentioned algorithm. Schellekens provided the asymptotic lower bound, and thus the asymptotic complexity class, of the running time of Mergesort.

3. Methodology

Given the set $\mathcal{C} = \left\{ f \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\}$, we use bicomplete quasi-metric $d_{\mathcal{C}}$ below to construct the complexity distance function

$$d_{\mathcal{C}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max \left\{ \frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right\}.$$

The complexity distance function classifies algorithms based on their asymptotic behavior. This function allows classification of algorithms based on their computational complexity. The complexity distance function is used to provide asymptotic upper bound of the divide and conquer recurrence

$$T(n) = \begin{cases} c & \text{if } n = 1, \\ aT\left(\frac{n}{b}\right) + h(n) & \text{if } n \in \mathbb{N}_b, \end{cases} \quad (3)$$

to solve complex problems by subdividing it into a subproblems and solving them separately then combine all subproblems solutions to give a global solution to the original problem.

Using the bicomplete quasi-metric space $(\mathcal{C}_{b,c}, d_{\mathcal{C}|\mathcal{C}_{b,c}})$, we associate the Divide and Conquer recurrence equation with the function $\Phi_T : \mathcal{C}_{b,c} \rightarrow \mathcal{C}_{b,c}$ to get

$$T(n) = \begin{cases} c & \text{if } n = 1, \\ \infty & \text{if } n \notin \mathbb{N}_b \text{ and } n > 1, \\ af\left(\frac{n}{b}\right) + h(n) & \text{otherwise,} \end{cases} \quad (4)$$

were the complexity function in $\mathcal{C}_{b,c}$ is a solution to the recurrence Equation if and only if it is a fixed point of the functional Φ_T . This gives

$$d_{\mathcal{C}|\mathcal{C}_{b,c}}(\Phi_T(f), \Phi_T(g)) \leq \frac{1}{a} d_{\mathcal{C}|\mathcal{C}_{b,c}}(f, g)$$

$\forall f, g \in \mathcal{C}_{b,c}$. This helps conclude that the functional $\Phi_T : \mathcal{C}_{b,c} \rightarrow \mathcal{C}_{b,c}$ has a unique fixed point and so the recurrence equation has a unique solution. The properties of metric is extended to b -dislocated metric by modifying symmetry:

$$d(x, z) \leq s(d(x, y) + d(y, z)), \quad s \geq 1.$$

The b -dislocated metric to be applied in computing involving digital evidence classification, where the similarity constraint:

$$d(x, z) \leq s(d(x, y) + d(y, z)), \quad s \geq 1$$

provides enhanced classification techniques. The fixed point theorem for b -dislocated metric spaces establishes the existence of unique fixed points under contraction conditions, contributing to classification models.

the b -metric space can be used to show that the distance between two elements x and z is bounded by a multiple of sum of the distance between x and y , and y and z . This property is used to measure similarity between elements of X and conclude that the distance between x and y is small then they are similar, otherwise they are dissimilar.

The Banach Fixed Point Theorem is also used to find the unique fixed points in recursive algorithms, formalized as:

$$d(f(x), f(y)) \leq cd(x, y), \quad 0 < c < 1$$

which guarantees convergence in iterative computational processes. This helps to see that the solution to the divide and conquer recurrence equation is unique regardless of the size of $n \in \mathbb{N}$ used to split the problem.

4. Application In Complexity Theory

We introduce a fixed point technique and discuss the complexity of algorithms by the use of partial quasi-metric. this will be used to asymptotic complexity analysis of those algorithms whose running time of computing leads to the Divide and Conquer recurrence [eq. \(3\)](#). Let us state the following proposition and theorem from [\[2, 18\]](#).

Proposition 4.1.

Suppose (X, p) is a partial metric space, then the function $d_p : X \times X \rightarrow \mathbb{R}^+$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ is a quasi-metric such that $\tau(p) = \tau(d_p)$.

Theorem 4.1.

Let (X, d) be a complete b -dislocated metric space, and let $T : X \rightarrow X$ satisfy

$$d(Tx, Ty) \geq \beta d(x, y) + \alpha d(x, y).$$

Then, T has a unique fixed point.

From [proposition 4.1](#), we have that the partial metric $p_{\mathcal{C}}$ induces the quasi-metric $d_{p_{\mathcal{C}}}$ on $\mathcal{C}$ given by

$$d_{p_{\mathcal{C}}}(f, g) = p_{\mathcal{C}}(f, g) - p_{\mathcal{C}}(f, f)$$

for all $f, g \in \mathcal{C}$. Further, it is easier to see that $p_{\mathcal{C}}(f, g) - p_{\mathcal{C}}(f, f) = d_{p_{\mathcal{C}}}(f, g)$.

Moreover, the order induced by the partial metric $p_{\mathcal{C}}, \leq_{p_{\mathcal{C}}}$, is exactly the pointwise order relation defined on $\mathcal{C}$. For $f, g \in \mathcal{C}$, we have that $f \leq_{p_{\mathcal{C}}} g \Leftrightarrow p_{\mathcal{C}}(f, g) = p_{\mathcal{C}}(f, f) \Leftrightarrow f(n) \leq_{\infty} g(n). \forall n \in \mathbb{N}$. Thus, $f \leq_{p_{\mathcal{C}}} g$ implies $f \in \mathcal{O}(g)$. Notice that, the completeness of the partial metric space $(\mathcal{C}, p_{\mathcal{C}})$ is guaranteed by the bicompleteness of the complexity space $(\mathcal{C}, d_{\mathcal{C}})$ and the following result from [\[15\]](#).

Proposition 4.2.

If (X, p) is a partial metric space, then the below assertions are equivalent:

1. (X, p) is complete.
2. (X, d_p) is bicomplete.

Since the partial metric space $(\mathcal{C}, p_{\mathcal{C}})$ is complete, it implies that the partial metric $(\mathcal{C}_{b,c}, p_{\mathcal{C}}|_{\mathcal{C}_{b,c}})$ is also complete. Now we discuss the complexity of Divide and Conquer algorithms whose running time of computing satisfy the Divide and Conquer recurrence [eq. \(3\)](#), let $s \in [0, 1]$ such that

$$p_{\mathcal{C}}|_{\mathcal{C}_{b,c}}(\Phi_T(f), \Phi_T(g)) \leq sp_{\mathcal{C}}|_{\mathcal{C}_{b,c}}(f, g)$$

$\forall f, g \in \mathcal{C}_{b,c}$ and Φ_T is a functional given by eq. (4). We consider the case of $s \in [0, 1]$ because clearly the case $s = 0$ gives a contradiction. Let $f, g \in \mathcal{C}_{b,c}$ such that $f(n) = 2c$ and $g(n) = 2(c+1)$ for all $n \in \mathbb{N}_b$. Then

$$p_{\mathcal{C}}|\mathcal{C}_{b,c}(\Phi_T(f), \Phi_T(g)) = \frac{1}{2c} + \sum_{n=2}^{\infty} 2^{-b^n} \max \left\{ \frac{1}{2ac + h(b^n)}, \frac{1}{2a(c+1) + h(b^n)} \right\}.$$

Moreover,

$$\begin{aligned} p_{\mathcal{C}}|\mathcal{C}_{b,c}(f, g) &= \frac{1}{2c} + \sum_{n=2}^{\infty} 2^{-b^n} \max \left\{ \frac{1}{f(b^n)}, \frac{1}{g(b^n)} \right\} \\ &\leq \sum_{n=2}^{\infty} 2^{-b^n} \max \left\{ \frac{1}{2c}, \frac{1}{2(c+1)} \right\} \\ &= \frac{1}{2c}. \end{aligned}$$

Applying our hypothesis, we get $\frac{1}{2c} = p_{\mathcal{C}}|\mathcal{C}_{b,c}(\Phi_T(f), \Phi_T(g)) \leq s p_{\mathcal{C}}|\mathcal{C}_{b,c}(f, g) = s \frac{1}{2c}$ deducing that $1 \leq s < 1$, hence a contradiction. Corollary [12, Corollary 2.4] cannot be applied to the asymptotic complexity analysis of those algorithms whose running time of computing leads to the Divide and Conquer recurrence eq. (3). Considering the partial metric $p_{\mathcal{C}}$ as a complexity distance instead of the original quasi-metric $d_{\mathcal{C}}$, a fixed point technique, different from the technique above is introduced to discuss the complexity of algorithms by the use of partial quasi-metric

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ aT(n-1) + h(n) & \text{if } n \geq 2 \end{cases} \quad (5)$$

where $c > 0$, $a \geq 1$ and $h \in \mathcal{C}$ with $h(n) <_{\infty} \infty$ for all $n \in \mathbb{N}$. Noticeably, the discussion of the asymptotic complexity of the Divide and Conquer algorithms introduced in section 3 can be carried out from the recurrence eq. (5). The running time of computing of the algorithms is the solution to the recurrence equation

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ aT(\frac{n}{b}) + h(n) & \text{if } n \in \mathbb{N}_b, \end{cases} \quad (6)$$

Evidently, the preceding Divide and Conquer recurrence equation can be retrieved as a particular case of the recurrence eq. (5). Thus, transforming the Divide and Conquer recurrence equation into

$$S(m) = \begin{cases} c & \text{if } m = 1 \\ aS(m-1) + r(m) & \text{if } m > 1 \end{cases} \quad (7)$$

where $S(m) = T(b^{m-1})$ and $r(m) = h(b^{m-1})$ for all $m \in \mathbb{N}$. Suppose the subset $\mathcal{C}_c$ of $\mathcal{C}$ is given by $\mathcal{C}_c = \{f \in \mathcal{C} : f(1) = c\}$.

We define the functional $\varphi_T : \mathcal{C}_c \longrightarrow \mathcal{C}_c$ by

$$\varphi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ af(n-1) + h(n) & \text{if } n \geq 2 \end{cases} \quad (8)$$

$\forall f \in \mathcal{C}_c$.

A complexity function in $\mathcal{C}_c$ is a solution to the recurrence eq. (5) if and only if it is a fixed point of the functional φ_T . Now to prove the existence and uniqueness of solution to the recurrence eq. (6), we use the following Lemma.

Lemma 4.1.

Suppose $\varphi_T : \mathcal{C}_c \longrightarrow \mathcal{C}_c$ is a functional given by eq. (7) and $f_{c,\infty} \in \mathcal{C}_c$ is the complexity function given by

$$f_{c,\infty}(n) = \begin{cases} c & \text{if } n = 1 \\ \infty & \text{if } n \geq 2 \end{cases}. \quad (9)$$

Then the following axioms should hold

1. φ_T is monotone
2. φ_T is continuous and p^s -continuous
3. $\varphi_T(f_{c,\infty}) \leq_{p_{\mathcal{C}_c}} f_{c,\infty}$.

Theorem 4.2.

The recurrence eq. (5) has a unique solution f_T in $\mathcal{C}_c$.

proof. The completeness of $(\mathcal{C}_c, p_{\mathcal{C}_c})$ and Lemma lemma 4.1 provide the conditions that φ_T has a fixed point in we obtain the existence of a fixed point of f in $\mathcal{C}_c$.

It remains to prove the uniqueness of the fixed point. To this end, assume that $f_T, g_T \in \mathcal{C}_c$ are solutions to the recurrence eq. (5). Next we will prove by induction over n that $f_T = g_T$. Since f_T, g_T are solutions to the recurrence eq. (5), we have that they are fixed points of the functional φ_T , i.e., $\varphi_T(f_T) = \varphi_T(g_T)$. Hence we have that $f_T(1) = \varphi_T(f_T)(1) = c = \varphi_T(g_T)(1) = g_T(1)$. Assuming that $\varphi(f_T)(n) = \varphi(g_T)(n)$ with $n > 1$. Then;

$$\begin{aligned} f_T(n+1) &= \varphi_T(f_T)(n+1) \\ &= af_T(n) + h(n+1) \\ &= ag_T(n) + h(n+1) \\ &= \varphi_T(g_T)(n+1) \\ &= g_T(n+1). \end{aligned}$$

Therefore, $f_T = g_T$ and so, φ_T has a unique fixed point in $\mathcal{C}_c$. Therefore the recurrence eq. (5) has a unique solution in $\mathcal{C}_c$. $\square$

5. Some Applications in Computer Forensics and Security

We demonstrates the theoretical foundation using b -dislocated metric spaces in similarity measurement applications. We first state the following definitions from [17, 18] and prove some related theorems.

Definition 5.1.

Given a nonempty set X and a mapping $d : X \times X \rightarrow [0, \infty]$ is called a b -dislocated metric if it satisfies the following axioms for any $x, y, z \in X$ and $s \geq 1$;

1. $d(x, y) = 0$ then $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b -dislocated metric space. The class of b -dislocated metric spaces is large than the class of dislocated metric spaces, this is because a b -dislocated metric is a dislocated metric when $s = 1$.

Definition 5.2.

A sequence (X_n) in a b -dislocated metric space (X, d) converges with d respect to $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. In this case, x is called the limit of (X_n) and we write $x_n \rightarrow x$.

Definition 5.3.

A sequence (X_n) in b -dislocated metric space (X, d) is called a Cauchy sequence if for $\epsilon > 0$, there exists $n_0 \in N$ such that $\forall m, n \geq n_0$ we have $d(x_m, x_n) < \epsilon$ or $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$.

Theorem 5.1.

Let (X, d) be a b -dislocated metric space, where $d : X \times X \rightarrow \mathbb{R}^+$ measures encryption key similarity. If the mapping $f : X \rightarrow X$ satisfies

$$d(f(x), f(y)) \leq d(x, y) \quad \text{for all } x, y \in X,$$

and (X, d) is complete, then f has a fixed point $x^* \in X$ such that $f(x^*) = x^*$. This fixed point models a stable and convergent cryptographic protocol.

proof. We proceed by constructing a Picard iteration. Choose any $x_0 \in X$ and define a sequence (x_n) by $x_{n+1} = f(x_n)$ for all $n \geq 0$. Since f is non-expansive, we have:

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) \leq d(x_n, x_{n+1}),$$

so the sequence $(d(x_n, x_{n+1}))$ is non-increasing and bounded below by 0. For $m > n$, using the b -relaxed triangle inequality:

$$d(x_n, x_m) \leq b \sum_{k=n}^{m-1} d(x_k, x_{k+1}).$$

Because $d(x_k, x_{k+1})$ decreases and tends to 0, the tail sum becomes arbitrarily small, proving that (x_n) is Cauchy. By completeness, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, x^*) = d(x^*, x^*)$.

Now observe:

$$d(f(x^*), x^*) \leq d(f(x^*), f(x_n)) + d(f(x_n), x_n),$$

where $d(f(x^*), f(x_n)) \leq d(x^*, x_n)$ by non-expansiveness, and $d(f(x_n), x_n) = d(x_{n+1}, x_n) \rightarrow 0$. So:

$$d(f(x^*), x^*) \leq d(x^*, x^*) + 0 = d(x^*, x^*).$$

Also, by the dislocated metric property, $d(x^*, x^*) \leq d(f(x^*), x^*)$, which implies $f(x^*) = x^*$. □

Elliptic curve cryptography utilizes structured distance functions to verify secure key generation in cryptographic protocols. We illustrate this in the next theorem.

Theorem 5.2.

Let (X, d) be a b -dislocated metric space modeling cryptographic keys. If a function $f : X \rightarrow \{0, 1\}$ satisfies

$$|f(x) - f(y)| \leq \alpha \cdot d(x, y)$$

for some $\alpha > 0$, then f is Lipschitz-continuous and robust to small perturbations in key data.

proof. Let $f : X \rightarrow \{0, 1\}$ be a function such that

$$|f(x) - f(y)| \leq \alpha \cdot d(x, y)$$

for all $x, y \in X$ and some constant $\alpha > 0$.

Since the codomain of f is discrete, the absolute difference $|f(x) - f(y)|$ is either 0 or 1. Thus, the inequality implies:

$$f(x) = f(y) \quad \text{whenever} \quad d(x, y) < \frac{1}{\alpha}.$$

This means that f is constant on d -balls of radius less than $1/\alpha$, i.e., small perturbations in the input (as measured by d) do not change the output of f . Therefore, f is Lipschitz-continuous with Lipschitz constant α . To verify Lipschitz continuity formally, consider any $x, y \in X$. Then by assumption:

$$|f(x) - f(y)| \leq \alpha \cdot d(x, y).$$

This is precisely the definition of Lipschitz continuity. Since f maps to $\{0, 1\}$ and is Lipschitz-continuous, any small change in the input (i.e., $d(x, y)$ small) ensures that $f(x) = f(y)$ or the change in output is bounded by $\alpha \cdot d(x, y) < 1$. But since the output is binary, this implies $f(x) = f(y)$ for sufficiently small $d(x, y)$, confirming robustness to perturbations. □

Theorem 5.3.

Suppose (X, d) is a complete b -dislocated metric space with a parameter $s \geq 1$, and T a continuous self mapping on X satisfying the following condition:

$$d(Tx, Ty) \geq \beta \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \alpha d(x, y) \tag{10}$$

$\forall x, y \in X, x \neq y$, then T has a unique fixed point.

Example 5.1.

Given $X = [0, 1]$ and a b -dislocated metric $d : X \times X \rightarrow \mathbb{R}^+$ defined as $d(x, y) = (x + y)^2$. Let the self mapping $T : X \rightarrow X$ defined as $Tx = 3x$. We notice that (X, d) is a complete b -dislocated metric space and $\forall x, y \in X$ we have;

$$d(Tx, Ty) = (3x + 3y)^2 = 9(x + y)^2 \geq \alpha d(x, y)$$

Hence, all conditions of [theorem 5.3](#) are satisfied and has a unique fixed point $x = 0$. In the next theorem, we use b -relaxed triangle inequality to measure similarity between elements of sets.

Theorem 5.4.

Let (X, d) be a b -dislocated metric space, where X is a set of digital evidence (images, documents, videos) and d is a distance function. Then, the b -dislocated metric can be used to measure similarity between elements of X .

proof. By definition, a b -dislocated metric space satisfies the following properties;

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq b(d(x, y) + d(y, z))$ (b -relaxed triangular inequality)

The b -relaxed triangle inequality ensures that the distance between two elements x and z is bounded by a multiple of the sum of the distances between x and y , and y and z . This property allows us to use the b -dislocated metric to measure similarity between elements of X . Specifically, if $d(x, y)$ is small, then x and y are similar. If $d(x, y)$ is large, then x and y are dissimilar.

The choice of b depends on the specific application and dataset being analysed. A smaller value of b will result in a more stringent similarity measure, while a larger value of b will result in a more relaxed similarity measure. Therefore, the b -dislocated metric can be used to measure similarity between digital evidence, such as images, documents, and videos. $\square$

We provide the following examples to illustrate the above theorem.

Example 5.2.

Suppose we have three images: x, y , and z . We want to measure the similarity between them using a b -dislocated metric space. Suppose $d(x, y) = 0.5$, $d(y, z) = 0.7$, $d(x, z) = 1.2$. Using the b -dislocated metric, the triangle inequality $d(x, z) \leq b(d(x, y) + d(y, z))$ holds by picking $b = 1.1$.

The inequality holds, so we can use the b -dislocated metric to measure similarity between the images.

Example 5.3.

Suppose we have three Videos: x, y , and z . The similarity between is measure using a b -dislocated metric space. If $d(x, y) = 0.2$, $d(y, z) = 0.6$, $d(x, z) = 1$ Using the b -dislocated metric, the triangle inequality $d(x, z) \leq b(d(x, y) + d(y, z))$ holds by choosing $b = 1.3$. The inequality holds, so we can use the b -dislocated metric to measure similarity between the videos.

Example 5.4.

Suppose we have three documents: x, y , and z . We measure the similarity between them using a b -dislocated metric space. Let's assume the distances between the documents are: $d(x, y) = 0.3$, $d(y, z) = 0.4$, $d(x, z) = 0.8$. Using the b -dislocated metric, the triangle inequality $d(x, z) \leq b(d(x, y) + d(y, z))$ holds by arbitrary choosing $b = 1.2$.

The inequality holds, so we can use the b -dislocated metric to measure similarity between the documents.

6. Conclusion

In this paper, we have applied metric spaces in computational complexity theory and used theorems to look at the fixed points in the recurrence equations and their uniqueness. We also looked at the application of metric spaces, particularly b -dislocated metric spaces in determining similarities in images, videos and documents. In short, this study presents a theorem-based approach to mathematically validate computational complexity. It highlights the foundational role of metric spaces in optimizing algorithms and lays out theoretical groundwork that supports practical applications. The latest publications [19, 21–25] have motivated us to publish this paper in this journal.

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8. Conflicts of Interest

The authors declare no conflicts of interest.

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