

Existence and uniqueness of solutions of system of linear equations and non-linear differential equations using modular b -metric integral type contractions.

Research Article

Anthony Katakwe^{*}, Danny Mukonda¹, Levy K Matindih¹, Tuesday Kafunda², Stanley Jere³, Kabuswa Davy Manyika⁴

¹ Department of Mathematics and Statistics, Mulungushi University, Kabwe-Zambia

² Department of Mathematics and Statistics, Eden University, Lusaka-Zambia

³ Department of Mathematics, Statistics and Actuarial Science, University of Zambia, Lusaka-Zambia

⁴ Department of Physics, Mulungushi University, Kabwe-Zambia

Volume 13, Issue 02

Received: 01 August 2025

Accepted (in revised version): 05 September 2025

Abstract: This study determines the existence and uniqueness of solutions of $n \times n$ systems of linear equations and non-linear differential equations (ODE's) using modular b -metric integral type contractions. Additionally, we analyze the fixed point theory using integral type contractions in modular b -metric spaces. The results of this research are expected to significantly contribute to the fixed point theory and stimulate further exploration in applied mathematics and theoretical computer science, paving a way for innovative methodologies and applications of modular b -metric integral type contractions.

MSC: 15A06 • 34A30 • 34A34

Keywords: Modular b -metric • Integral type contraction mapping • Unique fixed point • $n \times n$ system of linear equations • Non-linear differential equations(ODE's)

© 2025 The Author(s). This is an open access article under the CC BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/3.0/>).

1. Introduction

The aspects of modular metric spaces was first introduced by Chistyakov [1] in 2006. A modular metric w_λ is simply a metric d with a scalar parameter λ that allows the distance to vary with resolution. The generalization of a modular metric yields a modular b -metric v_λ which is a modular metric with a relaxation factor $s \geq 1$ on the triangle inequality. The work on modular metric expanded upon traditional metric spaces, providing a more adaptable method for examining convergence and continuity in mathematical analysis. After further research in 2010, Chistyakov [2] established fixed point theorems for both modular contractive and strongly modular contractive mappings, thereby extending the principles of classical fixed point theory in modular metric spaces.

^{*} Corresponding author.

E-mail address(es): anthonykatakwe110@gmail.com (Anthony Katakwe), dmukonda@mu.edu.zm (Danny Mukonda), lmattindih@mu.ac.zm (Levy K Matindih), ktuesday@edenuniversity.edu.zm (Tuesday Kafunda), sjere@unza.edu.zm (Stanley Jere), davymanyika@yahoo.com (Kabuswa Davy Manyika).

The work on contractions was first done by Banach [3] in complete metric spaces where the mapping brings points closer to each other within the same space. A self-mapping $T : X \rightarrow X$ is called a contraction mapping if it satisfies the condition $d(T(x), T(y)) \leq cd(x, y)$ for $c \in (0, 1)$ and $x, y \in X$. This foundational concept has since been further developed by many scholars in different perspectives [1, 2, 4–8]. Contractions in modular metric space and modular b -metric space also follow the same concept of bringing points closer to each other.

Branciari [4] defined integral type contractions as "Let (X, d) be a complete metric space, $k \in (0, 1)$, and let $f : X \rightarrow X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx, fy)} \varphi(t) dt \leq k \int_0^{d(x, y)} \varphi(t) dt \quad (1)$$

where $\varphi : [0, +\infty) \rightarrow [0, +\infty]$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$, nonnegative and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \rightarrow +\infty} f^n x = a$." This expanded upon the classical Banach contraction principle and established a general contractive condition of integral type within complete metric spaces. His research paved the way for further generalizations and applications of integral type contractions across various mathematical contexts.

Azadifar et al [5] proved the existence and uniqueness of a common fixed point for compatible mappings of integral type in modular metric spaces.

Ege and Alaca [7] expanded on the existing theories of modular metric spaces to define modular b -metric spaces, establishing new notions and proved fixed point theorems. Their work contributed significantly to the understanding and application of modular b -metric spaces in mathematical analysis and fixed point theory.

The theoretical foundations for the existence and uniqueness of solutions to systems of linear equations emerged through contributions of several centuries, with no single founder but pivotal advancements by key figures. The study of existence and uniqueness of solutions for non-linear differential equations emerged by several contributions in the 19th – 20th century mathematicians. Emile Picard [9] established the first comprehensive framework with the Picard-Lindelof Theorem, proving local existence and uniqueness for initial value problems.

2. Links to the literature

In this section, the literature review serves as a comprehensive exploration of existing research and scholarly works relevant to the subject matter of this research paper. This section aims to provide a contextual foundation, offering insights into the current state of knowledge, identifying gaps that will guide the discussions.

In 1991, E.Kreyszig [10] applied the fixed point theory to proving the existence and uniqueness of solutions for initial value problems (ODE's) on complete metric spaces. However, this idea has not been done in modular b -metric spaces.

In 2013, Azadifar et al [5] proved the existence and uniqueness of a common fixed point for compatible mappings of integral type in modular metric spaces. However, this knowledge of proving the existence and uniqueness of a common fixed point for compatible mappings of integral type contractions has not been done in modular b - metric spaces.

In 2018, Ege and Alaca [7] defined modular b -metric spaces. They established fixed point theorems and applied the results to determining the existence and uniqueness of solutions for $n \times n$ systems of linear equations. However, this application has not been done using integral type contractions in modular b -metric spaces.

In 2019, Parvaneh et al [11] generalized the F -contractive conditions to $\alpha - \nu$ -complete modular b -metric spaces and established some fixed point results in ordered modular b -metric spaces.

In 2022, Ozturk and Buyukkaya [12] specified a new C -class function endowed with altering distance and ultra altering distance function through generalized Ξ -contraction called $\mathcal{G}(\Sigma, \vartheta, \Xi)$ -contraction in modular b -metric spaces. They presented an application of their results to integral type contractions and graph theory.

In 2022, Ozturk and Buyukkaya [6] demonstrated some common fixed point theorems for diverse E -contractions involving generalized Proinov-simulation functions in modular b -metric spaces. They established an existence theorem for the solution of nonlinear integral equation.

In 2023, Ozturk et al [13] characterized a new class of simulation functions by extending the class of Z functions and established some common fixed point theorems in modular b -metric spaces. They applied the results of their research to finding the unique solution for integral equations.

In 2023, Buyukkaya et al [14] developed new E -type contractions called the Suzuki-Proinov generalized (α, Z_E^*) - type contractions mapping that included rational expressions by considering Proinov type functions and C_G -simulation functions within modular b -metric space.

In 2024, Buyukkaya and Ozturk [15] took the C_A simulation function and Proinov type function into account to come up with new contraction mapping called Suzuki Proinov $Z_{E^*}^{\mathcal{R}}(\alpha)$ -contraction. They also demonstrated a common fixed point theorem through the mappings endowed with triangular α -admissibility within modular b -metric spaces. Furthermore, they examined the existence of solutions to a class of functional equations emerging in dynamic programming.

The outline of this paper is as follows. Section 1, Introduction. Section 2, Literature review. Section 3, Methodology. In section 4, The application of the methodology including proofs and numerical examples are provided. In Section 5, The conclusion and suggested possible future research work are stated.

3. Methodology

In this section, we present the methods that guided our research paper to achieve the objectives of this study. Various journals, books, the internet and articles on modular b -metric spaces, integral type contractions [5] and Banach's contraction principle [16] were investigated to see how integral type contractions will hold in modular b -metric spaces. The Integral Type Contraction Theorem [4] "Let (X, d) be a complete metric space, $k \in (0, 1)$, and let $f : X \rightarrow X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx, fy)} \varphi(t) dt \leq k \int_0^{d(x, y)} \varphi(t) dt$$

where $\varphi : [0, +\infty) \rightarrow [0, +\infty]$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \rightarrow +\infty} f^n x = a$ " helped to analyze the fixed point theory using integral type contractions. By analyze we mean to prove the uniqueness of a common fixed point for self-mappings in certain theorems involving integral type contractions. We also applied the Extended Integral Type Contraction Theorem [7, Theorem 1] "Let X_ν^* be a ν -complete modular b -metric space, and let $T : X_\nu^* \rightarrow X_\nu^*$ be a ν -contraction mapping such that for all $x, y \in X_\nu^*$ and some $k \in [0, 1/s)$, with restriction $sk < 1$ for any $s \geq 1$, then the following integral type contraction condition holds

$$\int_0^{\nu_\lambda(Tx, Ty)} \varphi(t) dt \leq k \int_0^{\nu_\lambda(x, y)} \varphi(t) dt$$

where $\varphi : [0, \infty) \rightarrow [0, \infty]$ is a Lebesgue integrable mapping that is nonnegative and satisfies $\int_0^\epsilon \varphi(t) dt > 0$ for all $\epsilon > 0$. Assume there exists $x = x(\lambda) \in X_\nu^*$ such that $\nu_\lambda(x, Tx) < \infty$, then there is an element $\bar{x} \in X_\nu^*$ such that $x_n \rightarrow \bar{x}$, and $\bar{x}$ is a unique fixed point of T " to determine the existence and uniqueness of solutions of any $n \times n$ system of linear equations.

3.1. The following algorithm is used to determine the existence and uniqueness of solutions of any $n \times n$ system of linear equations

- (i) Check if the system is diagonally dominant. That is
for each row i , $|a_{ij}|_{i=j} > \sum_{i \neq j} |a_{ij}|$ and in the case of the system

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} = b_1 \\ a_{21} & a_{22} & \dots & a_{2n} = b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} = b_n \end{pmatrix}$$

we have $|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$, $|a_{22}| > |a_{21}| + |a_{23}| + \dots + |a_{2n}|$ and so on.

- (ii) Define the fixed point mapping T . We define the fixed point mapping of T by making the first variable in the row the subject, then go to the next row and make the second variable the subject and so on. For the system

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} = b_1 \\ a_{21} & a_{22} & \dots & a_{2n} = b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} = b_n \end{pmatrix}$$

$a_{11} = b_1 - a_{12} - \dots - a_{1n}$, $a_{22} = b_2 - a_{21} - \dots - a_{2n}$ and so on so that $T(a_{11}, a_{22}, \dots, a_{nn})$.

- (iii) We prove that the mapping T is a contraction using the Jacobian matrix of T by calculating the matrix norm (∞ -norm). The matrix norm (∞ -norm) gives us the contraction factor k that is

$$\|T\|_\infty = k.$$

By the Banach contraction principle, $k \in (0, 1)$ for contraction to be guaranteed so as to have a unique fixed point for the contraction of T .

(iv) We verify that T is an integral type contraction. That is

$$\int_0^{v_\lambda(Tx, Ty)} \phi(t) dt \leq k \int_0^{v_\lambda(x, y)} \phi(t) dt.$$

(v) We check for the uniqueness of the solution. That is

$$(1 - k) \int_0^{v_\lambda(\bar{x}, y)} \phi(t) dt \leq 0$$

which implies that $(1 - k) \geq 0$ and since $\phi(t)$ is positive then $v_\lambda(\bar{x}, y) = 0 \implies \bar{x} = y$.

3.2. The following algorithm is used to determine the existence and uniqueness of solutions of certain non-linear differential equations

(i) Convert the non-linear differential equation problem to a fixed point problem by rewriting it into an integral equation using

$$Ty(t) = g(t) + \int_0^t g(s, y(s)) ds$$

(ii) Verify the integral type contraction condition by using the Lipschitz condition

$$|g(x) - g(y)| \leq L|x - y| \quad \text{for all } x, y \in \mathbb{R}$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a mapping and $L < 1$ is the contraction factor.

(iii) Apply the Banach contraction principle "Let (X, d) be a non-empty complete metric space. A mapping $f : X \rightarrow X$ is called a contraction mapping if there exists a constant $c \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$ holds and f has a unique fixed point $a \in X$ such that for every $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$."

4. Application

4.1. Theorems that analyze the existence and uniqueness of a fixed point using integral type contractions.

In this section, we presented Theorems that help us demonstrate how to analyze the fixed point theorem using integral type contractions based entirely on [5].

Definition 4.1.

Let X_v be a modular b -metric space induced by modular b -metric v . Two self-mappings T, h of X_v are called v -compatible if $v_\lambda(Thx_n, hTx_n) \rightarrow 0$, whenever $\{x_n\}_{n=1}^\infty$ is a sequence in X_v such that $hx_n \rightarrow z$ and $Tx_n \rightarrow z$ for some point $z \in X_v$ and for $\lambda > 0$.

Theorem 4.1.

Let X_v be a complete modular b -metric space. Suppose that $c, k, l \in \mathbb{R}^+, c > l$ and $T, h : X_v \rightarrow X_v$ are two v -compatible mappings such that $T(X_v) \subseteq h(X_v)$ and

$$\int_0^{v_\lambda^{\frac{1}{c}}(Tx, Ty)} \varphi(t) dt \leq k \int_0^{v_\lambda^{\frac{1}{l}}(hx, hy)} \varphi(t) dt \quad (2)$$

for some $k \in (0, 1)$ and for $\lambda > 0$, where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a Lebesgue integrable function, nonnegative and for all $\epsilon > 0$,

$$\int_0^\epsilon \varphi(t) dt > 0. \quad (3)$$

If one of T or h is continuous, then there exists a unique common fixed point of T and h .

Example 4.1.

Let $X_v = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $v_\lambda(x, y) = \frac{|x-y|}{\lambda}$ for any $\lambda > 0$. Define the mapping $T, h : X_v \rightarrow X_v$ by

$$T(x) = \begin{cases} \frac{1}{n+4} & \text{if } x = \frac{1}{n} \\ 0 & \text{if otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \\ 0 & \text{if otherwise} \end{cases}$$

Using the theorem above, the hypothesis holds for $\varphi(t) = 2t$, where $t > 0$ and $c = 2, l = 1$ and $k = \frac{1}{4}$ since $T(\frac{1}{n}) = \frac{1}{n+4}$, $T(0) = 0$ and $h(\frac{1}{n}) = \frac{1}{n+1}$, $h(0) = 0$ which implies that T, h are v -compatible with a unique common fixed point 0. Also $T(X_v) \subseteq h(X_v)$ since $T(\frac{1}{n}) = \frac{1}{n+4} = h(\frac{1}{n+3})$.

Proof. Let's assume x to be an arbitrary point of X_v and generate inductively the sequence $\{Tx_n\}_{n=1}^{\infty}$ as $Tx_n = hx_{n+1}$ for each n and $x_1 = x$ since $T(X_v) \subseteq h(X_v)$. (2) shows that for each integer $n \geq 1$ and for every $\lambda > 0$ we have

$$\begin{aligned} \int_0^{v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n)} \varphi(t) dt &\leq k \int_0^{v_{\frac{\lambda}{l}}(hx_{n+1}, hx_n)} \varphi(t) dt \\ &\leq k \int_0^{v_{\frac{\lambda}{c}}(Tx_n, Tx_{n-1})} \varphi(t) dt \\ &\leq k^2 \int_0^{v_{\frac{\lambda}{l}}(hx_n, hx_{n-1})} \varphi(t) dt. \end{aligned}$$

By the principle of mathematical induction, we can easily show that

$$\int_0^{v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n)} \varphi(t) dt \leq k^n \int_0^{v_{\frac{\lambda}{l}}(Tx, x)} \varphi(t) dt.$$

To demonstrate the inequality using mathematical induction, we follow the standard procedure of mathematical induction.

Assuming we have a recursive relationship for $v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n)$, then

$$v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n) \leq k v_{\frac{\lambda}{c}}(Tx_n, Tx_{n-1}).$$

Suppose

$$C = \int_0^{v_{\frac{\lambda}{l}}(Tx, x)} \varphi(t) dt$$

so that

$$\int_0^{v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n)} \varphi(t) dt \leq k^n C.$$

Step 1: We start with the base case for $n = 1$,

$$\int_0^{v_{\frac{\lambda}{c}}(Tx_2, Tx_1)} \varphi(t) dt \leq k^1 C.$$

Step 2: Assuming that the statement holds for some integer $n = k$,

$$\int_0^{v_{\frac{\lambda}{c}}(Tx_{k+1}, Tx_k)} \varphi(t) dt \leq k^k C.$$

Step 3: We now show for $n = k + 1$ so that

$$\int_0^{v_{\frac{\lambda}{c}}(Tx_{k+2}, Tx_{k+1})} \varphi(t) dt \leq k^{k+1} C.$$

From the recursive relationship obtained earlier we have

$$\begin{aligned} \int_0^{v_{\frac{\lambda}{c}}(Tx_{k+2}, Tx_{k+1})} \varphi(t) dt &\leq k \int_0^{v_{\frac{\lambda}{c}}(Tx_{k+1}, Tx_k)} \varphi(t) dt \\ &\leq k(k^k C) \\ &= k^{k+1} C. \end{aligned}$$

This shows that if it holds for $n = k$ then it also holds for $n = k + 1$ and so

$$\int_0^{v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n)} \varphi(t) dt \leq k^n \int_0^{v_{\frac{\lambda}{l}}(Tx, x)} \varphi(t) dt$$

is true for all integers $n \geq 1$. By taking the limit as $n \rightarrow \infty$, yields

$$\lim_{n \rightarrow \infty} \int_0^{v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n)} \varphi(t) dt \leq 0.$$

Thus by (3) we have,

$$\lim_{n \rightarrow \infty} v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n) = 0.$$

We now show that $\{Tx_n\}_{n=1}^\infty$ is Cauchy. For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $v_{\frac{\lambda}{c}}(Tx_{n+1}, Tx_n) < \frac{\epsilon}{c}$ for all $n \in \mathbb{N}$ with $n \geq n_0$ and $\lambda > 0$. Without loss of generality, suppose $m, n \in \mathbb{N}$ and $m > n$. Take note that for $\frac{\lambda}{c(m-n)} > 0$, there exists $n \frac{\lambda}{c(m-n)} \in \mathbb{N}$ such that

$$v_{\frac{\lambda}{c(m-n)}}(Tx_{n+1}, Tx_n) < \frac{\epsilon}{c(m-n)} \text{ for all } n \geq n \frac{\lambda}{c(m-n)}. \text{ We then have}$$

$$\begin{aligned} v_{\frac{\lambda}{c}}(Tx_n, Tx_m) &\leq v_{\frac{\lambda}{c(m-n)}}(Tx_n, Tx_{n+1}) + v_{\frac{\lambda}{c(m-n)}}(Tx_{n+1}, Tx_{n+2}) + \dots + v_{\frac{\lambda}{c(m-n)}}(Tx_{m-1}, Tx_m) \\ &< \frac{\epsilon}{c(m-n)} + \frac{\epsilon}{c(m-n)} + \dots + \frac{\epsilon}{c(m-n)} \\ &= (m-n) \frac{\epsilon}{c(m-n)} \\ &= \frac{\epsilon}{c} \end{aligned}$$

for every $n, m \geq n \frac{\lambda}{c(m-n)}$ and this shows that $\{Tx_n\}_{n=1}^\infty$ is a Cauchy sequence. Since X_v is complete, there exists a point $z \in X_v$ such that $v_{\frac{\lambda}{c}}(Tx_n, z) \rightarrow 0$ as $n \rightarrow \infty$. Suppose T is continuous, then $T^2x_n \rightarrow Tz$ and $Thx_n \rightarrow Tz$. By the v -compatibility of X_v , we have $v_{\lambda}(hTx_n, Thx_n) \rightarrow 0$ as $n \rightarrow \infty$ for $\lambda > 0$. Also $hTx_n \rightarrow Tz$ since $v_{\lambda}(hTx_n, Tz) \leq v_{\frac{\lambda}{2}}(hTx_n, Thx_n) + v_{\frac{\lambda}{2}}(Thx_n, Tz)$. In this case, we now prove that z is a common fixed point of T and h . By (2) we have,

$$\int_0^{v_{\frac{\lambda}{c}}(T^2x_n, Tx_n)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{l}}(hTx_n, hx_n)} \varphi(t) dt.$$

As $n \rightarrow \infty$, this yields

$$\int_0^{v_{\frac{\lambda}{c}}(Tz, z)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{l}}(Tz, z)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{c}}(Tz, z)} \varphi(t) dt$$

which implies that $v_{\frac{\lambda}{c}}(Tz, z) = 0$ for $\lambda > 0$ and so $Tz = z$. Since $T(X_v) \subseteq h(X_v)$ there exists a point z_1 , such that $z = Tz = hz_1$. By (2) we have,

$$\int_0^{v_{\frac{\lambda}{c}}(T^2x_n, Tz_1)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{l}}(hTx_n, hz_1)} \varphi(t) dt.$$

As $n \rightarrow \infty$ yields

$$\int_0^{v_{\frac{\lambda}{c}}(Tz, Tz_1)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{l}}(Tz, hz_1)} \varphi(t) dt,$$

and so

$$\int_0^{v_{\frac{\lambda}{c}}(z, Tz_1)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{l}}(z, hz_1)} \varphi(t) dt \leq k \int_0^{v_{\frac{\lambda}{l}}(z, z)} \varphi(t) dt.$$

Thus $z = Tz_1 = hz_1$ and also $hz = hTz_1 = Thz_1 = Tz = z$. If one considers h to be continuous (instead of T), then by similar argument as above, one can prove that $hz = Tz = z$. Lastly if z and w are two arbitrary common fixed points of T and h respectively, then we have

$$\begin{aligned} \int_0^{v_{\frac{\lambda}{c}}(z, w)} \varphi(t) dt &= \int_0^{v_{\frac{\lambda}{c}}(Tz, Tw)} \varphi(t) dt \\ &\leq k \int_0^{v_{\frac{\lambda}{l}}(hz, hw)} \varphi(t) dt \\ &\leq k \int_0^{v_{\frac{\lambda}{c}}(z, w)} \varphi(t) dt, \end{aligned}$$

which implies that $v_{\frac{\lambda}{c}}(z, w) = 0$ for $\lambda > 0$ and $z = w$. □

Theorem 4.2.

[7, Compare Theorem 1] Let X_v^* be a v -complete modular b -metric space, and let $T : X_v^* \rightarrow X_v^*$ be a v -contraction mapping such that for all $x, y \in X_v^*$ and some $k \in [0, 1/s)$, with restriction $sk < 1$ for any $s \geq 1$, then the following integral type contraction condition holds

$$\int_0^{v_{\lambda}(Tx, Ty)} \varphi(t) dt \leq k \int_0^{v_{\lambda}(x, y)} \varphi(t) dt$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable mapping that is nonnegative and satisfies $\int_0^{\epsilon} \varphi(t) dt > 0$ for all $\epsilon > 0$. Assume there exists $x = x(\lambda) \in X_v^*$ such that $v_{\lambda}(x, Tx) < \infty$, then there is an element $\bar{x} \in X_v^*$ such that $x_n \rightarrow \bar{x}$, and $\bar{x}$ is a unique fixed point of T .

Proof. Let $x_0 \in X_v^*$ and define the sequence $(x_n)_{n \in \mathbb{N}}$ iteratively by $x_n = Tx_{n-1} = T^n x_0$ for $n \in \mathbb{N}$. The integral type contraction condition gives us

$$\int_0^{\nu_\lambda(x_n, x_{n+1})} \phi(t) dt = \int_0^{\nu_\lambda(Tx_{n-1}, Tx_n)} \phi(t) dt \leq k \int_0^{\nu_\lambda(x_{n-1}, x_n)} \phi(t) dt \quad \text{for } \lambda > 0.$$

Then for $x_n = T^n x_0$, we get

$$\int_0^{\nu_\lambda(x_n, x_{n+1})} \phi(t) dt \leq k \int_0^{\nu_\lambda(x_{n-1}, x_n)} \phi(t) dt \leq \dots \leq k^n \int_0^{\nu_\lambda(x_0, x_1)} \phi(t) dt$$

which simply implies

$$\int_0^{\nu_\lambda(T^n x_0, T^n x_1)} \phi(t) dt \leq k^n \int_0^{\nu_\lambda(x_0, x_1)} \phi(t) dt.$$

We now need to show that $(x_n)_{n \in \mathbb{N}}$ is Cauchy. To show that $(x_n)_{n \in \mathbb{N}}$ is ν -Cauchy, let's consider $m, n > 0$ with $m > n$. Using integral type contraction property and the properties of ϕ we deduce

$$\int_0^{\nu_\lambda(x_n, x_m)} \phi(t) dt \leq s \int_0^{\nu_{\frac{\lambda}{m-n}}(x_n, x_{n+1})} \phi(t) dt + \dots + s^{m-n} \int_0^{\nu_{\frac{\lambda}{m-n}}(x_{m-1}, x_m)} \phi(t) dt.$$

Applying the contraction property gives us

$$\int_0^{\nu_\lambda(x_n, x_m)} \phi(t) dt \leq sk^n \int_0^{\nu_{\frac{\lambda}{m-n}}(x_0, x_1)} \phi(t) dt + \dots + s^{m-n} k^{m-1} \int_0^{\nu_{\frac{\lambda}{m-n}}(x_0, x_1)} \phi(t) dt.$$

Factorizing out the common terms simplifies to

$$\int_0^{\nu_\lambda(x_n, x_m)} \phi(t) dt = \int_0^{\nu_{\frac{\lambda}{m-n}}(x_0, x_1)} \phi(t) dt \cdot sk^n [1 + sk + (sk)^2 + \dots + (sk)^{m-n-1}].$$

If $n, m \rightarrow \infty$, the terms $(sk)^{m-n-1} \rightarrow 0$ and $sk^n \rightarrow 0$ (since $sk < 1$), and so we have

$$\lim_{n, m \rightarrow \infty} \int_0^{\nu_\lambda(x_n, x_m)} \phi(t) dt = 0.$$

Since ϕ is positive and summable means that

$$\lim_{n, m \rightarrow \infty} \nu_\lambda(x_n, x_m) = 0.$$

Thus this shows that $(x_n)_{n \in \mathbb{N}}$ is a ν -Cauchy sequence in X_v^* .

By the ν -completeness of X_v^* , the sequence $(x_n)_{n \in \mathbb{N}}$ is ν -convergent and converges to $\bar{x} \in X_v^*$. To show that $\bar{x}$ is a fixed point of T , let's consider the following

$$\int_0^{\nu_\lambda(T\bar{x}, \bar{x})} \phi(t) dt \leq s \left[\int_0^{\nu_{\frac{\lambda}{2}}(T\bar{x}, x_n)} \phi(t) dt + \int_0^{\nu_{\frac{\lambda}{2}}(x_n, \bar{x})} \phi(t) dt \right].$$

Considering the contraction property only on the first term on the right of the inequality results in

$$\int_0^{\nu_\lambda(T\bar{x}, \bar{x})} \phi(t) dt \leq s \left[k \int_0^{\nu_{\frac{\lambda}{2}}(\bar{x}, x_{n-1})} \phi(t) dt + \int_0^{\nu_{\frac{\lambda}{2}}(x_n, \bar{x})} \phi(t) dt \right].$$

If $n \rightarrow \infty$, both terms $\int_0^{\nu_{\frac{\lambda}{2}}(\bar{x}, x_{n-1})} \phi(t) dt$ and $\int_0^{\nu_{\frac{\lambda}{2}}(x_n, \bar{x})} \phi(t) dt$ tend to zero which means that

$$\int_0^{\nu_\lambda(T\bar{x}, \bar{x})} \phi(t) dt = 0.$$

Since ϕ is positive implies that $\nu_\lambda(T\bar{x}, \bar{x}) = 0$ and so $T\bar{x} = \bar{x}$.

Lastly, we need to prove the uniqueness of the fixed point $\bar{x}$. To prove the uniqueness of the fixed point $\bar{x}$, let's assume y is another fixed point of T . By integral type contraction condition

$$\int_0^{\nu_\lambda(\bar{x}, y)} \phi(t) dt = \int_0^{\nu_\lambda(T\bar{x}, Ty)} \phi(t) dt \leq k \int_0^{\nu_\lambda(\bar{x}, y)} \phi(t) dt.$$

By subtracting $k \int_0^{\nu_\lambda(\bar{x}, y)} \phi(t) dt$ from both sides we obtain

$$(1 - k) \int_0^{\nu_\lambda(\bar{x}, y)} \phi(t) dt \leq 0.$$

Since $1 - k > 0$ and ϕ is positive implies $\nu_\lambda(\bar{x}, y) = 0$ and so $\bar{x} = y$. □

4.2. Application to a system of linear equations

Integral type contractions can be used to determine whether a system of linear equations has a unique solution or not. In the following, we show how to determine the existence and uniqueness of solutions for any $n \times n$ system of linear equations based on the work of [7].

The following examples demonstrate how to determine the existence and uniqueness of a solution for an $n \times n$ system of linear equations.

Example 4.2.

[7, Compare Theorem 4] Let $X = \mathcal{R}^n$ be a modular b -metric space with modular b -metric

$$v_\lambda(x, y) = \frac{d(x, y)}{\lambda},$$

where $x, y \in X$ and

$$d(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

If

$$\sum_{i=1}^n |\alpha_{ij}| \leq \alpha < 1 \quad \text{for } j = 1, 2, \dots, n,$$

then the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

of n linear equations in n unknowns has a unique solution.

Proof. To solve this problem, we consider the following steps:

Step 1: We check if the system is diagonally dominant.

Assuming that $|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$, $|a_{22}| > |a_{21}| + |a_{23}| + \dots + |a_{2n}|$ and so on holds, we can say that the system is diagonally dominant.

Step 2: We define the fixed point mapping T .

$$T(x_1, \dots, x_n) = \left(\frac{b_1 - a_{12}x_2 - \dots - a_{1n}x_n}{a_{11}}, \dots, \frac{b_n - a_{n1}x_1 - \dots - a_{nn-1}x_{n-1}}{a_{nn}} \right).$$

Step 3: We prove that T is a contraction using the Jacobian matrix of T . In our case, we cannot find the actual matrix norm (∞ -norm) but $\|T\|_\infty = k = \alpha$ is the contraction factor. Considering the distance function of T

$$\begin{aligned} v_\lambda(Tx, Ty) &= \frac{1}{\lambda} d(Tx, Ty) \\ &= \frac{1}{\lambda} \sum_{i=1}^n \left| \sum_{j=1}^n \alpha_{ij}(x_j - y_j) \right| \\ &\leq \frac{1}{\lambda} \sum_{i=1}^n \sum_{j=1}^n |\alpha_{ij}| |x_j - y_j| \\ &= \frac{1}{\lambda} \sum_{j=1}^n \sum_{i=1}^n |\alpha_{ij}| |x_j - y_j| \\ &\leq \frac{1}{\lambda} \sum_{j=1}^n \alpha |x_j - y_j| \quad \left(\text{since } \sum_{i=1}^n |\alpha_{ij}| \leq \alpha \right) \\ &= \frac{\alpha}{\lambda} \sum_{j=1}^n |x_j - y_j| \\ &= \alpha v_\lambda(x, y). \end{aligned}$$

By the Banach contraction principle, $k = \alpha \in (0, 1)$ for contraction to be guaranteed so as to have a unique fixed point for the contraction of T .

Step 4: We verify the integral type contraction condition.

By using integral type contraction property where $\phi(t) = 1$ we have

$$\begin{aligned} \int_0^{\nu_\lambda(Tx, Tx)} \phi(t) dt &= \int_0^{\nu_\lambda(Tx, Tx)} 1 dt \\ &= \nu_\lambda(Tx, Tx) \\ &\leq \alpha \nu_\lambda(x, x) \\ &= \alpha \int_0^{\nu_\lambda(x, x)} 1 dt \\ &= \alpha \int_0^{\nu_\lambda(x, x)} \phi(t) dt \end{aligned}$$

and we conclude that indeed T is an integral type contraction mapping.

Step 5: We now check for the uniqueness of the solution.

Assuming the fixed points $\bar{x}$ and y

$$\int_0^{|\bar{x}-y|} \phi(t) dt \leq \alpha \int_0^{|\bar{x}-y|} \phi(t) dt$$

subtracting $\alpha \int_0^{|\bar{x}-y|} \phi(t) dt$ from both sides of the inequality we get

$$(1 - \alpha) \int_0^{|\bar{x}-y|} \phi(t) dt \leq 0.$$

Since $(1 - \alpha) \geq 0$ and $\phi(t)$ is positive, then $|\bar{x} - y| = 0 \implies \bar{x} = y$. This shows that the system of linear equations has a unique solution $\bar{x}$. $\square$

Example 4.3.

Let $X_v^* = \mathbb{R}^3$ be a complete modular b -metric space equipped with a modular b -metric such that $\nu_\lambda(x, y) = \frac{d(x, y)}{\lambda}$ where $\lambda = 1$ and $d(x, y) = \sum_{i=1}^n |x_i - y_i|$ for all $x, y \in X_v^*$.

If

$$\sum_{i=1}^n |\alpha_{ij}| \leq \alpha < 1 \quad \text{for } j = 1, 2, \dots, n,$$

then the system of linear equations

$$\begin{cases} 4x + y + z = 6 \\ x + 5y + z = 7 \\ x + y + 6z = 8 \end{cases}$$

of 3 linear equations in 3 unknowns has a unique solution.

Proof. To solve this problem, we consider the following steps:

Step 1: We check if the system is diagonally dominant. Since

$$|4| > |1| + |1| = 2, |5| > |1| + |1| = 2, |6| > |1| + |1| = 2.$$

The system is diagonally dominant.

Step 2: We define the fixed point mapping T .

$$T(x, y, z) = \left(\frac{6 - y - z}{4}, \frac{7 - x - z}{5}, \frac{8 - x - y}{6} \right).$$

Step 3: We prove that T is a contraction using the Jacobian matrix of T .

Given the operator $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where

$$T(x, y, z) = \left(\frac{6 - y - z}{4}, \frac{7 - x - z}{5}, \frac{8 - x - y}{6} \right)$$

we let

$$T_1(x, y, z) = \left(\frac{6 - y - z}{4} \right), T_2(x, y, z) = \left(\frac{7 - x - z}{5} \right), T_3(x, y, z) = \left(\frac{8 - x - y}{6} \right).$$

The partial derivatives give us the following

$$\frac{\partial T_1}{\partial x} = 0, \frac{\partial T_1}{\partial y} = -\frac{1}{4}, \frac{\partial T_1}{\partial z} = -\frac{1}{4}$$

$$\frac{\partial T_2}{\partial x} = -\frac{1}{5}, \frac{\partial T_2}{\partial y} = 0, \frac{\partial T_2}{\partial z} = -\frac{1}{5}$$

$$\frac{\partial T_3}{\partial x} = -\frac{1}{6}, \frac{\partial T_3}{\partial y} = -\frac{1}{6}, \frac{\partial T_3}{\partial z} = 0$$

now,

$$T = \begin{pmatrix} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} & \frac{\partial T_1}{\partial z} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} & \frac{\partial T_2}{\partial z} \\ \frac{\partial T_3}{\partial x} & \frac{\partial T_3}{\partial y} & \frac{\partial T_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{5} & 0 & -\frac{1}{5} \\ -\frac{1}{6} & -\frac{1}{6} & 0 \end{pmatrix}.$$

The induced matrix norm(∞ – norm) gives the contraction factor

$$\|T\|_{\infty} = \max\left\{\frac{1}{2}, \frac{2}{5}, \frac{1}{3}\right\} = \frac{1}{2}.$$

Since $\|T\|_{\infty} = \frac{1}{2} < 1$, T is a contraction and by the Banach contraction principle has a unique solution in $\mathbb{R}^3$. Therefore, $|Tx - Ty| \leq k|x - y|$ with $k = \frac{1}{2} < 1$ guarantees that the contraction of T has a unique fixed point.

Step 4: We verify the integral contraction condition.

Picking $\phi(t) = 1$,

$$\begin{aligned} \int_0^{\nu_{\lambda}(Tx, Ty)} 1 dt &= \int_0^{|Tx - Ty|} 1 dt \\ &= |Tx - Ty| \\ &\leq k|x - y| \\ &= k \int_0^{\nu_{\lambda}(x, y)} 1 dt. \end{aligned}$$

Step 5: We now check for uniqueness of the solution

Assuming the fixed points $\bar{x}$ and y

$$\int_0^{|\bar{x} - y|} \phi(t) dt \leq \frac{1}{2} \int_0^{|\bar{x} - y|} \phi(t) dt$$

subtracting $\frac{1}{2} \int_0^{|\bar{x} - y|} \phi(t) dt$ from both sides of the inequality we get

$$\left(1 - \frac{1}{2}\right) \int_0^{|\bar{x} - y|} \phi(t) dt \leq 0.$$

Since $(1 - \frac{1}{2}) > 0$ and $\phi(t)$ is positive, then $|\bar{x} - y| = 0 \implies \bar{x} = y$. This shows that the system of linear equations has a unique solution $\bar{x}$. $\square$

4.3. Application to non-linear differential equations

In this section, we show how to apply integral type contractions to determining the existence and uniqueness of the solution for non-linear differential equations and the work is entirely based on [9], [10] and [17].

The following examples demonstrates how to determine the existence and uniqueness of solutions for non-linear differential equations.

Example 4.4.

Let $X_{\nu}^* = C([0, 1], \mathbb{R})$ be a ν -complete modular b -metric space of continuous functions on $[0, 1]$ equipped with the modular b -metric

$$\nu_{\lambda}(x, y) = \frac{1}{\lambda} \int_0^1 |x(t) - y(t)| dt \quad \text{where } \lambda > 0.$$

If the initial value problem (IVP) $y'(t) = f(t, y(t))$, $y(0) = 0$, $t \in [0, 1]$, where $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[0, 1]$ and satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq L|x - y| \quad \text{with } L < 1,$$

then the IVP has a unique solution.

Proof. Step 1. Convert the IVP to a fixed point problem by rewriting it as an integral equation

$$Ty(t) = y(t) + \int_0^t f(u, y(u)) du$$

where $y(t)$ is the solution which is the fixed point of the mapping T .

Step 2. Verify the integral type contraction condition.

Choosing $\phi(t) = 1$, we need to show that T satisfies

$$\int_0^{v_\lambda(Tx, Ty)} \phi(t) dt \leq k \int_0^{v_\lambda(x, y)} \phi(t) dt \quad \text{for } k = L < 1.$$

The distance function $v_\lambda(Tx, Ty)$ gives

$$\begin{aligned} v_\lambda(Tx, Ty) &= \frac{1}{\lambda} \int_0^1 |Tx(t) - Ty(t)| dt \\ &= \frac{1}{\lambda} \int_0^1 \left| \int_0^t (f(u, x(u)) - f(u, y(u))) du \right| dt. \end{aligned}$$

Applying the Lipschitz condition

$$|Tx(t) - Ty(t)| \leq \int_0^t L|x(u) - y(u)| du.$$

Integrating over the interval $t \in [0, 1]$ gives

$$\int_0^1 |Tx(t) - Ty(t)| dt \leq L \int_0^1 \int_0^t |x(u) - y(u)| du dt.$$

Changing the order of integration for $t \geq u, u \in [0, 1]$ and $t \in [u, 1]$ gives us

$$\begin{aligned} \int_0^1 |Tx(t) - Ty(t)| dt &\leq L \int_0^1 |x(u) - y(u)| \left(\int_u^1 dt \right) du \\ &= L \int_0^1 |x(u) - y(u)| (1 - u) du. \end{aligned}$$

Since $1 - u \leq 1$, we have

$$\int_0^1 |Tx(t) - Ty(t)| dt \leq L \int_0^1 |x(u) - y(u)| du.$$

By multiplying with $\frac{1}{\lambda}$

$$\begin{aligned} \frac{1}{\lambda} \int_0^1 |Tx(t) - Ty(t)| dt &= v_\lambda(Tx, Ty) \\ &\leq \frac{L}{\lambda} \int_0^1 |x(u) - y(u)| du \\ &= Lv_\lambda(x, y). \end{aligned}$$

Using the integral type contraction condition since $\phi(t) = 1$,

$$\begin{aligned} \int_0^{v_\lambda(Tx, Ty)} 1 dt &= v_\lambda(Tx, Ty) \\ &\leq k \int_0^{v_\lambda(x, y)} 1 dt \\ &= kv_\lambda(x, y). \end{aligned}$$

Then $k = L < 1$ and T is an integral type contraction.

Step 3. Apply the Banach contraction principle.

Since X_ν^* is ν -complete and T satisfies the integral type contraction condition, then by Banach contraction principle there exists a unique fixed point $\bar{x} \in X_\nu^*$ such that $T\bar{x} = \bar{x}$ and this fixed point $\bar{x}$ is the unique solution to the initial value problem (IVP). $\square$

Example 4.5.

Let $X_b^* = C([0, T], \mathbb{R})$ be a ν -complete modular b -metric space of continuous functions on $[0, T]$ equipped with the modular b -metric

$$\nu_\lambda(x, y) = \frac{1}{\lambda} \int_0^1 |x(t) - y(t)| dt \quad \text{where } \lambda > 0.$$

If the initial value problem (IVP),

$$\frac{dg}{dt} = -\alpha g(t), g(t) = g_0 \quad \text{at } t = 0 \implies g(0) = g_0 \quad \text{for all } \alpha > 0$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[0, T]$ and satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \leq L|x - y| \quad \text{with } L < 1,$$

then the IVP has a unique solution.

Proof. Step 1. Convert the IVP to a fixed point problem by rewriting it as an integral equation

$$\begin{aligned} Ty(t) &= g(t) + \int_0^t -\alpha g(s, y(s)) ds \\ &= g(t) - \alpha \int_0^t g(s, y(s)) ds. \end{aligned}$$

Step 2. Verify the integral type contraction condition.

Considering the distance between these two mappings

$$(Tg_1)(t) - (Tg_2)(t) = -\alpha \int_0^t (g_1(s) - g_2(s)) ds.$$

The supremum norm over $[0, T]$ yields

$$\|Tg_1 - Tg_2\| = \sup_{t \in [0, T]} |-\alpha \int_0^t (g_1(s) - g_2(s)) ds|$$

By integral norm inequality we have

$$|\int_0^t g(s) ds| \leq \int_0^t |g(s)| ds$$

and so

$$\begin{aligned} \|Tg_1 - Tg_2\| &= \|-\alpha \int_0^t (g_1(s) - g_2(s)) ds\| \\ &= \|\alpha(Tg_1 - Tg_2)\| \end{aligned}$$

thus,

$$\|Tg_1 - Tg_2\| \leq \alpha T \|g_1 - g_2\|.$$

Now assuming the Lipschitz condition holds then $\alpha T = L < 1$ for $\alpha > 0$.

Choosing $\phi(t) = 1$, we need to show that T satisfies

$$\int_0^{\nu_\lambda(Tx, Ty)} \phi(t) dt \leq k \int_0^{\nu_\lambda(x, y)} \phi(t) dt \quad \text{for } k = L < 1.$$

Computing the distance function $\nu_\lambda(Tx, Ty)$,

$$\begin{aligned} \nu_\lambda(Tx, Ty) &= \frac{1}{\lambda} \int_0^1 |Tx(t) - Ty(t)| dt \\ &= \frac{\alpha}{\lambda} \int_0^1 \left| \int_0^t (g(s, x(s)) - g(s, y(s))) ds \right| dt. \end{aligned}$$

Applying the Lipschitz condition

$$|Tx(t) - Ty(t)| \leq \int_0^t L|x(s) - y(s)| ds.$$

Integrating over the interval $t \in [0, T]$ gives us

$$\int_0^1 |Tx(t) - Ty(t)| dt \leq L \int_0^1 \int_0^t |x(s) - y(s)| ds dt.$$

Changing the order of integration for $t \geq s, s \in [0, T]$ and $t \in [s, T]$ gives us

$$\begin{aligned} \int_0^1 |Tx(t) - Ty(t)| dt &\leq L \int_0^1 |x(s) - y(s)| \left(\int_s^T dt \right) ds \\ &= L \int_0^1 |x(s) - y(s)| (T - s) ds. \end{aligned}$$

Since $T - s \leq T \leq 1$, we get

$$\int_0^1 |Tx(t) - Ty(t)| dt \leq L \int_0^1 |x(s) - y(s)| ds.$$

By multiplying with $\frac{1}{\lambda}$

$$\begin{aligned} \frac{1}{\lambda} \int_0^1 |Tx(t) - Ty(t)| dt &= v_\lambda(Tx, Ty) \\ &\leq \frac{L}{\lambda} \int_0^1 |x(s) - y(s)| ds \\ &= Lv_\lambda(x, y). \end{aligned}$$

Since $\phi(t) = 1$, substituting into the integral type contraction condition gives us

$$\begin{aligned} \int_0^{v_\lambda(Tx, Ty)} 1 \cdot dt &= v_\lambda(Tx, Ty) \\ &\leq k \int_0^{v_\lambda(x, y)} 1 \cdot dt \\ &= kv_\lambda(x, y). \end{aligned}$$

Then $k = L < 1$, and T is an integral type contraction.

Step 3. Apply the Banach contraction principle.

Since X_v^* is v -complete and T satisfies the integral type contraction condition, then by Banach contraction principle there exists a unique fixed point $\bar{x} \in X_v^*$ such that $T\bar{x} = \bar{x}$ and this fixed point $\bar{x}$ is the unique solution to the initial value problem (IVP).

□

5. Conclusion

This paper presents a successful study of the existence and uniqueness of solutions of systems of linear equations and non-linear differential equations using modular b -metric integral type contractions. We analyzed the fixed point theorem by using integral type contractions to determine the existence and uniqueness of a common fixed point in certain theorems presented. In the last part of our work, we successfully applied modular b -metric integral type contractions in solving problems such as determining the existence and uniqueness of the solution of any $n \times n$ system of linear equations and differential equations such as non-linear differential equations (ODE's). It is important to note that our latest publications [8, 18] have motivated us to publish this paper in this journal.

6. Acknowledgements

The authors are thankful to the Department of Mathematics and Statistics and Mulungushi University for using their resources. Many thanks also go to colleagues who helped in making comments.

7. Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] V. Chistyakov, "Metric modulars and their application," in *Doklady Mathematics*, vol. 73. Springer, 2006, pp. 32–35.

- [2] V. V. Chistyakov, “Modular metric spaces, ii: Application to superposition operators,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 15–30, 2010.
- [3] A. Latif, “Banach contraction principle and its generalizations,” *Topics in fixed point theory*, pp. 33–64, 2014.
- [4] A. Branciari et al., “A fixed point theorem for mappings satisfying a general contractive condition of integral type,” *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 9, pp. 531–536, 2002.
- [5] B. Azadifar, G. Sadeghi, R. Saadati, and C. Park, “Integral type contractions in modular metric spaces,” *Journal of Inequalities and Applications*, vol. 2013, pp. 1–14, 2013.
- [6] A. Büyükkaya and M. Öztürk, “Some common fixed point results for z - contractions in modular b-metric spaces,” *Topological Algebra and its Applications*, vol. 10, no. 1, pp. 196–215, 2022.
- [7] M. E. Ege and C. Alaca, “Some results for modular b-metric spaces and an application to system of linear equations,” *Azerbaijan Journal of Mathematics*, vol. 8, no. 1, pp. 3–14, 2018.
- [8] K. C. Mukonda D, Kafunda T and jere S, “Analysis on the heat and mass transfer in boundary layer non-newtonian nanofluid past a vertically stretching porous plate with chemical reaction, variable magnetic field and variable thermal conductivity,” *International of Journal of Advanced Applied Mathematics and Mechanics*, vol. 11, no. 4, pp. 1–14, 2024.
- [9] É. Picard, “Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives,” *Journal de Mathématiques pures et appliquées*, vol. 6, pp. 145–210, 1890.
- [10] E. Kreyszig, *Introductory functional analysis with applications*. John Wiley & Sons, 1991.
- [11] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki, and H. Aydi, “Fixed point results for generalized f-contractions in modular b-metric spaces with applications,” *Mathematics*, vol. 7, no. 10, p. 887, 2019.
- [12] M. Öztürk and A. Büyükkaya, “On some fixed point theorems for $\mathcal{G}(\Sigma, \vartheta, \xi)$ - contractions in modular b - metric spaces,” *Fundamental Journal of Mathematics and Applications*, vol. 5, no. 4, pp. 210–227, 2022.
- [13] M. Ozturk, F. Golkarmanesh, A. Buyukkaya, and V. Parvaneh, “Generalized almost simulative $\hat{Z}_{\psi}^{\theta}$ -contraction- contraction mappings in modular b- metric spaces,” *Journal of Mathematical Extension*, vol. 17, 2023.
- [14] A. Büyükkaya, A. Fulg, and M. Öztürk, “On generalized suzuki-proinov type $(\alpha, z * e)$ - contractions in modular b- metric spaces,” *Filomat*, vol. 37, no. 4, pp. 1207–1222, 2023.
- [15] A. Büyükkaya and M. Öztürk, “On suzuki-proinov type contractions in modular b -metric spaces with an application,” *Communications in Advanced Mathematical Sciences*, vol. 7, no. 1, pp. 27–41, 2024.
- [16] S. Banach, “Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,” *Fundamenta mathematicae*, vol. 3, no. 1, pp. 133–181, 1922.
- [17] M. A. Johnson, “Mathematical methods lecture notes,” Lecture notes, University of Notre Dame, 2015, chapter 4: Introduction to Fixed Point Methods.
- [18] T. Kafunda, M. Danny, and N. Mwamba, “Hydromagnetic nanofluid flow with lorentz force, viscous dissipation, dufour effect, first-order chemical reaction and unsteadiness,” *Journal of Innovative Applied Mathematics and Computational Sciences*, vol. 4, no. 2, pp. 137–152, 2024.

Submit your manuscript to IJAAMM and benefit from:

- Rigorous peer review
- Immediate publication on acceptance
- Open access: Articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► editor.ijaamm@gmail.com